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Potential symmetries and new solutions of a simplified model for reacting mixtures

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Abstract. In this paper, we investigate potential symmetries of a simplified model for reacting mixtures. We find new similarity reductions and wider class of solutions through this approach. Further, we explore an invertible mapping which linearizes the reacting mixture model.

1. Introduction

For the past two decades group theoretical methods have been applied to solve a wide range of problems and to explore many physically interesting solutions [1–9] of nonlinear phenomena. Moreover, in order to arrive at new solutions of partial differential equations (PDEs) which are not obtainable through the classical Lie algorithm several extensions and modifications of the original Lie group method have been proposed in different contexts [10].

Recently, Bluman and Kumei introduced another method to find a new class of symmetries for a PDE system, say $R\{t, x, \boldsymbol{u}\}$, in the case that at least one of the PDEs can be written in a conserved form [1,11]. If we introduce the potential variables $\boldsymbol{v}$, for the PDE system $R\{t, x, \boldsymbol{u}\}$ written in a conserved form, as further unknown functions we obtain another system (*auxiliary system*), say $S\{t, x, \boldsymbol{u}, \boldsymbol{v}\}$. After finding the Lie point symmetries of the auxiliary system $S\{t, x, \boldsymbol{u}, \boldsymbol{v}\}$ let us suppose that any one of the components, of the infinitesimal operators, corresponding to the variables t, x and $\boldsymbol{u}$ depends explicitly on the potential variables $\boldsymbol{v}$. Hence these infinitesimal operators are not projectable to the space $\{t, x, \boldsymbol{u}\}$ and these local symmetries of $S\{t, x, \boldsymbol{u}, \boldsymbol{v}\}$ induce non-local symmetries for the original system $R\{t, x, \boldsymbol{u}\}$. These kind of non-local symmetries which are neither Lie point nor Lie–Bäcklund symmetries have been called, by Bluman and Kumei, *potential symmetries*.

Consequently, intense research has been started to study the concept of potential symmetries both from the theoretical as well as practical (applications) point of view [11–13]. Concerning the applications of the potential symmetry approach several investigations have been made to explore new physically interesting solutions which are not obtainable through Lie point symmetries for certain nonlinear PDEs. For example, Sophocleous has considered the nonlinear diffusion–convection types of equations and examined potential symmetries and associated similarity solutions [14]. Similarly, Gandarias has carried out a detailed analysis on the potential symmetries of a porous medium equation and constructed some interesting new solutions [15].

For our purposes it is worth noticing that a potential symmetry of the system $R\{t, x, \boldsymbol{u}\}$ is a point symmetry of the system $S\{t, x, \boldsymbol{u}, \boldsymbol{v}\}$ so one can extend the uses of point symmetries to the potential symmetries. In particular:

(i) Invariant solutions of $S\{t, x, \boldsymbol{u}, \boldsymbol{v}\}$ give solutions of $R\{t, x, \boldsymbol{u}\}$. These solutions, of course, are not invariant solutions for any local symmetry admitted by $R\{t, x, \boldsymbol{u}\}$.

(ii) If $S\{t, x, \boldsymbol{u}, \boldsymbol{v}\}$ is linearizable then $R\{t, x, \boldsymbol{u}\}$ is also linearizable. Concerning this last point we stress that quite often when a potential symmetry exists its infinitesimal operator is infinite-dimensional and allows one to linearize the system, whereas the Lie algebra of $R\{t, x, \boldsymbol{u}\}$ is finite-dimensional.

Motivated by the above facts, in this paper, we wish to study the potential symmetries and look for new solutions of the following simplified model for a binary unimolecular reacting exothermic mixture

$$\begin{aligned} &u_t + \left(\frac{u^2}{2} - \alpha q\right)_x - \beta u_{xx} = 0 \\ &q_x = \gamma q f(u) \end{aligned} \tag{1.1}$$

where u is a lumped variable with some features of pressure or temperature, q is the mass fraction of the reactant, $\alpha > 0$ is the heat released by the reaction, β is a lumped diffusion coefficient, γ is the reaction rate and $f(u) \geqslant 0$ is a structure function derived through asymptotic considerations. The coordinate, x, is not a spatial coordinate but a generalized coordinate representing a spacetime of the reaction zone.

The system (1.1) is the specialization to the one-dimensional case of model equations derived by Rosales and Majda [16] (see also [17] and references therein) by a systematic application of the method of multiple scaling and matched asymptotic expansions to the Navier–Stokes equation for a unimolecular exothermic reacting mixture. The success of this model and of similar simplified qualitative models rests on their ability to produce in a transparent fashion analogues of the complex phenomena which they model. The Navier–Stokes equations, for a reacting mixture, involve many peculiar effects produced through the nonlinear interaction of chemical and fluid mechanical phenomena. This model is motivated by an attempt to understand the interactions of strongly nonlinear sound waves, chemistry, and diffusion.

However, as far as we know, there are no studies concerning the search for exact analytical solutions, although some attempts have been undertaken to get numerical solutions in some special cases (see, e.g., [18]). Only recently, the invariance properties of equation (1.1) under a one-parameter Lie group of infinitesimal transformations have been studied by Rigano and Torrisi [19]. They gave the complete classification with respect to the function $f(u)$ and obtained classes of invariant solutions in some cases.

We study, here, the system (1.1) using the potential symmetries approach to look for new classes of exact solutions. Interestingly, the results of our investigation tell us that the system (1.1) admits potential symmetries for a particular choice of $f(u)$. Furthermore, we show that it is only under the existence of potential symmetries that the similarity reduced ordinary differential equations (ODEs) take a linear form whereas under the usual Lie point symmetries the reduced ODEs turn out to be nonlinear. Using the symmetries for this particular choice of $f(u)$ we unearth an explicit transformation which linearizes the system (1.1).

The plan of the paper is as follows. In section 2 we study the potential symmetries for the reacting mixture model. In section 3 we investigate similarity reductions and explore new solutions. In section 4 we show additional point symmetries for the system (1.1). In section 5, we present the invertible mapping and linearize the reaction mixture model. Finally, we present the conclusions in section 6.

2. Potential symmetries

To determine the potential symmetries of equations (1.1) let us rewrite them in a potential form so that

$$\begin{aligned} v_x &= u \\ v_t &= -\frac{u^2}{2} + \alpha q + \beta u_x \\ q_x &= \gamma q f(u). \end{aligned} \tag{2.1}$$

The invariance of equation (2.1) under the one-parameter Lie group of infinitesimal transformations,

$$\begin{aligned} x &\longrightarrow X = x + \varepsilon\xi_1(t, x, u, v, q) \\ t &\longrightarrow T = t + \varepsilon\xi_2(t, x, u, v, q) \\ u &\longrightarrow U = u + \varepsilon\phi_1(t, x, u, v, q) \\ v &\longrightarrow V = v + \varepsilon\phi_2(t, x, u, v, q) \\ q &\longrightarrow Q = q + \varepsilon\phi_3(t, x, u, v, q) \qquad \varepsilon \ll 1 \end{aligned} \tag{2.2}$$

leads, by applying the well known techniques (see [1–9]), to the following results:

$$\begin{aligned} \xi_1 &= b_1(t) + \frac{\dot{a}(t)}{2}x \qquad \xi_2 = a(t) \\ \phi_1 &= c_{1x}(t, x)\mathrm{e}^{\frac{v}{2\beta}} + \frac{\ddot{a}(t)}{2}x + \dot{b}_1(t) + \left[\frac{c_1(t, x)}{2\beta}\mathrm{e}^{\frac{v}{2\beta}} - \frac{\dot{a}(t)}{2}\right]u \\ \phi_2 &= c_1(t, x)\mathrm{e}^{\frac{v}{2\beta}} + \frac{\ddot{a}(t)}{4}x^2 + \dot{b}_1(t)x + e_1(t) \\ \phi_3 &= \frac{1}{\alpha}[c_{1t}(t, x) - \beta c_{1xx}(t, x)]\mathrm{e}^{\frac{v}{2\beta}} + \frac{\dddot{a}(t)}{4\alpha}x^2 + \frac{\ddot{b}_1(t)}{\alpha}x \\ &\quad + \frac{\dot{e}_1(t)}{\alpha} - \frac{\beta\ddot{a}(t)}{2\alpha} + \left[\frac{c_1(t, x)}{2\beta}\mathrm{e}^{\frac{v}{2\beta}} - \dot{a}(t)\right]q \end{aligned} \tag{2.3}$$

provided that $\xi_1, \xi_2, \phi_1, \phi_3$, satisfy identically the following classifying condition:

$$\phi_{3x} + u\phi_{3v} + \gamma q f(u)(\phi_{3q} - \xi_{1x} - u\xi_{1v}) - \gamma f(u)\phi_3 - \gamma q f'(u)\phi_1 = 0. \tag{2.4}$$

In equations (2.3) a, b_1, e_1 are arbitrary functions of t and c_1 is an arbitrary function of the variables t and x. Moreover in equations (2.3) and equations (2.4) subscripts denote the partial derivatives with respect to the variable x, t, v and q respectively and the dots denote the total derivatives with respect to t.

At this point we wish to recall that a Lie point symmetry of equations (2.1) is said to be a potential symmetry of the original system, equations (1.1), only if at least one of the infinitesimal components ξ_1, ξ_2, ϕ_1, ϕ_3 admits explicitly the potential variable v. Since our aim is to find the *potential symmetries* for equations (1.1) we try to explore the form of $f(u)$ such that the function $c_1(t, x)$ in the infinitesimals is non-zero. By keeping this idea in mind and inserting equations (2.3) in equation (2.4) we find that only for the form of

$$f(u) = \frac{u}{2\beta\gamma} + k \tag{2.5}$$

with k arbitrary constant, does the system (2.1) admit potential symmetries with non-zero form for $c_1(t,x)$. The associated infinitesimal symmetries for the above form of $f(u)$ turn out to be

$$\begin{aligned}
&\xi_1 = b_1 - k\beta\gamma a_1 t + \frac{a_1}{2}x \qquad \xi_2 = a_0 + a_1 t\\
&\phi_1 = c_{1x}(t,x)\mathrm{e}^{\frac{v}{2\beta}} + \left[\frac{1}{2\beta}c_1(t,x)\mathrm{e}^{\frac{v}{2\beta}} - \frac{a_1}{2}\right]u - k\beta\gamma a_1\\
&\phi_2 = c_1(t,x)\mathrm{e}^{\frac{v}{2\beta}} + c_2 - k\beta\gamma a_1 x\\
&\phi_3 = \frac{1}{\alpha}\chi(t)\mathrm{e}^{\frac{v}{2\beta}+k\gamma x} + \left[\frac{1}{2\beta}c_1(t,x)\mathrm{e}^{\frac{v}{2\beta}} - a_1\right]q
\end{aligned} \tag{2.6}$$

where a_0, a_1, b_1, c_2 are arbitrary constants and $c_1(t,x)$ is a solution of the following inhomogeneous linear heat equation:

$$c_{1t}(t,x) - \beta c_{1xx}(t,x) = \chi(t)\mathrm{e}^{k\gamma x} \tag{2.7}$$

with $\chi(t)$ an arbitrary function of t.

The associated Lie algebra is infinite-dimensional and it is spanned by

$$\begin{aligned}
&X_1 = \partial_x \qquad X_2 = \partial_t \qquad X_3 = \partial_v\\
&X_4 = (x - 2k\beta\gamma t)\partial_x + 2t\partial_t - (u + 2k\beta\gamma)\partial_u - 2k\beta\gamma x\partial_v - 2q\partial_q\\
&X_{c_1(t,x)} = \left[c_{1x}(t,x) + \frac{u}{2\beta}c_1(t,x)\right]\mathrm{e}^{\frac{v}{2\beta}}\partial_u + c_1(t,x)\mathrm{e}^{\frac{v}{2\beta}}\partial_v + \frac{q}{2\beta}c_1(t,x)\mathrm{e}^{\frac{v}{2\beta}}\partial_q\\
&X_{\chi(t)} = \frac{1}{\alpha}\chi(t)\mathrm{e}^{\frac{v}{2\beta}+k\gamma x}\partial_q.
\end{aligned} \tag{2.8}$$

The infinitesimal operators X_1, X_2, X_3, X_4 are projectable to the space $\{t, x, u, q\}$ and are the point symmetries of system (1.1). The infinitesimal operators $X_{c_1(t,x)}$ and $X_{\chi(t)}$ are not projectable. They are the infinitesimal operators of the desired potential symmetries and generate an infinite-parameter group (subgroup) of transformations.

3. Similarity reductions

In this section we investigate similarity reductions associated with the infinitesimal symmetries (2.6). The characteristic equations associated with the infinitesimal symmetries can be written as

$$\begin{aligned}
\frac{\mathrm{d}x}{b_1 - k\beta\gamma a_1 t + \frac{a_1}{2}x} &= \frac{\mathrm{d}t}{a_0 + a_1 t}\\
&= \frac{\mathrm{d}u}{c_{1x}(t,x)\mathrm{e}^{\frac{v}{2\beta}} + \left[\frac{1}{2\beta}c_1(t,x)\mathrm{e}^{\frac{v}{2\beta}} - \frac{a_1}{2}\right]u - k\beta\gamma a_1}\\
&= \frac{\mathrm{d}v}{c_1(t,x)\mathrm{e}^{\frac{v}{2\beta}} + c_2 - k\beta\gamma a_1 x}\\
&= \frac{\mathrm{d}q}{\frac{1}{\alpha}\chi(t)\mathrm{e}^{\frac{v}{2\beta}+k\gamma x} + \left[\frac{1}{2\beta}c_1(t,x)\mathrm{e}^{\frac{v}{2\beta}} - a_1\right]q}.
\end{aligned} \tag{3.1}$$

We solve equations (3.1) for the following two different cases.

Case 1: $a_1 = 0$. In this case we get the similarity variables of the form (with $a_0 = 1$, for simplicity)

$$
\begin{aligned}
z &= x - b_1 t \\
w_1 &= u\left[w_2 - \frac{1}{2\beta}\int \bar{c}_1(t,z)\mathrm{e}^{\frac{c_2 t}{2\beta}}\,\mathrm{d}t\right] - \int c_{1z}(t,z)\mathrm{e}^{\frac{c_2 t}{2\beta}}\,\mathrm{d}t \\
w_2 &= \frac{1}{2\beta}\int \bar{c}_1(t,z)\mathrm{e}^{\frac{c_2 t}{2\beta}}\,\mathrm{d}t + \mathrm{e}^{\frac{(c_2 t - v)}{2\beta}} \\
w_3 &= q\left[w_2 - \frac{1}{2\beta}\int \bar{c}_1(t,z)\mathrm{e}^{\frac{c_2 t}{2\beta}}\,\mathrm{d}t\right] - \frac{1}{\alpha}\int \chi(t)\mathrm{e}^{\gamma k x + \frac{c_2 t}{2\beta}}\,\mathrm{d}t
\end{aligned}
\tag{3.2}
$$

where c_1 is a solution of equation (2.7) along characteristic lines. Under this similarity transformation equation (2.1) can be reduced to the following system of ODEs:

$$2\beta w_2' + w_1 = 0 \tag{3.3a}$$

$$2\beta b_1 w_2' - \beta w_1' + c_2 w_2 - \alpha w_3 = 0 \tag{3.3b}$$

$$w_3' - k\gamma w_3 = 0 \tag{3.3c}$$

where prime denotes differentiation with respect to z.

Case 2: $a_1 \neq 0$. In this case because of the complexity of the calculations we work with a special class of solutions of equation (2.7):

$$c_1(t,x) = \left[\mathrm{e}^{\beta\gamma^2 k^2 t}\int \chi(t)\mathrm{e}^{-\beta\gamma^2 k^2 t}\,\mathrm{d}t + I_1\mathrm{e}^{\beta\gamma^2 k^2 t}\right]\mathrm{e}^{\gamma k x} \tag{3.4}$$

where I_1 is an integration constant. Substituting this form in the characteristic equation (3.1) we get the similarity variables of the form

$$
\begin{aligned}
z &= \frac{x}{(a_0 + a_1 t)^{(1/2)}} + \frac{2(b_1 + k\beta\gamma a_0)}{a_1(a_0 + a_1 t)^{(1/2)}} + \frac{2k\beta\gamma(a_0 + a_1 t)^{(1/2)}}{a_1} \\
w_1 &= 2\beta(u + 2k\beta\gamma)(a_0 + a_1 t)^{\frac{1}{2}}\left[w_2 - \frac{\mathrm{e}^{\frac{-2k\gamma(b_1 + 2k\beta\gamma a_0)}{a_1}}}{2\beta}\int [I_1 + G(t)](a_0 + a_1 t)^{A-1}\mathrm{d}t\right] \\
w_2 &= (a_0 + a_1 t)^A \mathrm{e}^{(k^2\beta\gamma^2 t - k\gamma z(a_0 + a_1 t)^{(1/2)} - \frac{v}{2\beta})} \\
&\quad + \frac{\mathrm{e}^{\frac{-2k\gamma(b_1 + 2k\beta\gamma a_0)}{a_1}}}{2\beta}\int [I_1 + G(t)](a_0 + a_1 t)^{A-1}\mathrm{d}t \\
w_3 &= 2\beta(a_0 + a_1 t)\left[w_2 - \frac{\mathrm{e}^{\frac{-2k\gamma(b_1 + 2k\beta\gamma a_0)}{a_1}}}{2\beta}\int [I_1 + G(t)](a_0 + a_1 t)^{A-1}\mathrm{d}t\right]q \\
&\quad - \frac{2\beta}{\alpha}\mathrm{e}^{\frac{-2k\gamma(b_1 + 2k\beta\gamma a_0)}{a_1}}\int \dot{G}(t)(a_0 + a_1 t)^A\,\mathrm{d}t
\end{aligned}
\tag{3.5}
$$

where

$$G(t) = \int \chi(t)\mathrm{e}^{-\beta\gamma^2 k^2 t}\,\mathrm{d}t \qquad A = \frac{c_2 + 2k\beta\gamma(b_1 + k\beta\gamma a_0)}{2\beta a_1}. \tag{3.6}$$

Using the similarity transformations (3.5) one can rewrite equation (2.1) to obtain

$$4\beta^2 w_2' + w_1 = 0 \tag{3.7a}$$

$$z w_2' - \frac{1}{2\beta a_1}w_1' + 2A w_2 - \frac{\alpha}{2\beta^2 a_1}w_3 = 0 \tag{3.7b}$$

$$w_3' = 0 \tag{3.7c}$$

where prime denotes differentiation with respect to the variable z.

It is interesting to note that in both the cases $a_1 = 0$ (see equation (3.3)) and $a_1 \neq 0$ (see equation (3.7)) the reduced ODE system is a linear one.

3.1. New solutions

In this section we present new solutions associated with the reacting mixtures model (1.1) considering the different subcases from the previous section.

Case 1: $a_1 = 0$. Solving equation (3.3) we get a solution of the form

$$\begin{aligned} w_1 &= -2\beta\left[m_1 I_2 \mathrm{e}^{m_1 z} + m_2 I_3 \mathrm{e}^{m_2 z} + \frac{\alpha I_4 \gamma k}{(c_2 + 2\gamma^2 k^2 \beta^2 - 2b_1\beta\gamma k)}\mathrm{e}^{\gamma k z}\right] \\ w_2 &= I_2 \mathrm{e}^{m_1 z} + I_3 \mathrm{e}^{m_2 z} + \frac{\alpha I_4}{(c_2 + 2\gamma^2 k^2 \beta^2 - 2b_1\beta\gamma k)}\mathrm{e}^{\gamma k z} \\ w_3 &= I_4 \mathrm{e}^{\gamma k z} \end{aligned} \tag{3.8}$$

where I_2, I_3 and I_4 are integration constants while m_1 and m_2 are solutions of the algebraic equation

$$\beta m^2 + b_1 m + \frac{c_2}{2\beta} = 0. \tag{3.9}$$

Rewriting (3.8) in terms of the old variables we get a solution of the form

$$\begin{aligned} u &= \frac{\int c_{1x}(t,x)\mathrm{e}^{\frac{c_2 t}{2\beta}}\,\mathrm{d}t}{H(t,x)} - \frac{2\beta}{H(t,x)}\bigg[m_1 I_2 \mathrm{e}^{m_1(x-b_1 t)} + m_2 I_3 \mathrm{e}^{m_2(x-b_1 t)} \\ &\qquad + \frac{\alpha I_4 \gamma k \mathrm{e}^{\gamma k(x-b_1 t)}}{(c_2 + 2\gamma^2 k^2 \beta^2 - 2b_1\beta\gamma k)}\bigg] \\ v &= \log\left[\frac{\mathrm{e}^{\frac{c_2 t}{2\beta}}}{H(t,x)}\right]^{2\beta} \\ q &= \frac{\mathrm{e}^{\gamma k x}\int \chi(t)\mathrm{e}^{\frac{c_2 t}{2\beta}}\,\mathrm{d}t}{\alpha H(t,x)} + \frac{I_4 \mathrm{e}^{\gamma k(x-b_1 t)}}{H(t,x)} \end{aligned} \tag{3.10}$$

where

$$\begin{aligned} H(t,x) &= I_2 \mathrm{e}^{m_1(x-b_1 t)} + I_3 \mathrm{e}^{m_2(x-b_1 t)} + \frac{\alpha I_4 \mathrm{e}^{\gamma k(x-b_1 t)}}{(c_2 + 2\gamma^2 k^2 \beta^2 - 2b_1\beta\gamma k)} \\ &\qquad - \frac{1}{2\beta}\int c_1(t,x)\mathrm{e}^{\frac{c_2 t}{2\beta}}\,\mathrm{d}t. \end{aligned} \tag{3.11}$$

It is easy to verify that the functions u and q given by (3.10) are solutions of the system (1.1).

The presence of a solution $c_1(t, x)$ of equation (2.7) in (3.10) gives us more possibilities to satisfy, quite general, suitable auxiliary (i.e. initial or boundary) conditions. In other words, by appropriately choosing the form for $c_1(t, x)$ and the arbitrary constants, one could get some physically interesting solutions for the reacting mixtures model (1.1).

Case 2: $a_1 \neq 0$. Solving equation (3.7c) we get $w_3 = \text{constant} = I_5$. However, to investigate the nature of solutions we proceed with the two cases, that is, $A = 0$ and $A \neq 0$.

Subcase 2(a): $A = 0$. In this case, from (3.7*b*), we get,

$$w_1' + \left(\frac{a_1 z}{2\beta}\right) w_1 + \frac{\alpha I_5}{\beta} = 0. \tag{3.12}$$

A general solution for equation (3.12) can be written as

$$w_1 = \frac{-\alpha I_5}{\beta} \mathrm{e}^{(\frac{-a_1 z^2}{4\beta})} \int \mathrm{e}^{(\frac{a_1 z^2}{4\beta})}\, \mathrm{d}z + I_6 \mathrm{e}^{(\frac{-a_1 z^2}{4\beta})} \tag{3.13}$$

where I_6 is an integration constant. By using equation (3.7*a*), we get

$$w_2 = \frac{\alpha I_5}{4\beta^3} \int \mathrm{e}^{(\frac{-a_1 z^2}{4\beta})} \left(\int \mathrm{e}^{(\frac{a_1 z^2}{4\beta})}\, \mathrm{d}z \right) \mathrm{d}z - \frac{I_6}{4\beta^2} \int \mathrm{e}^{(\frac{-a_1 z^2}{4\beta})}\, \mathrm{d}z + I_7 \tag{3.14}$$

where I_7 is an integration constant. Substituting the expressions for w_1, w_2 and w_3 in the equations (3.5) one gets

$$\begin{aligned} u &= \frac{P(x,t)}{2\beta(a_0 + a_1 t)^{(1/2)} Q(x,t)} - 2k\beta\gamma \\ v &= \log \left[\frac{\mathrm{e}^{(k^2\beta\gamma^2 t - k\gamma z (a_0 + a_1 t)^{(1/2)})}}{Q(x,t)} \right]^{2\beta} \\ q &= \frac{I_5}{2\beta(a_0 + a_1 t) Q(x,t)} + \frac{\mathrm{e}^{\frac{-2k\gamma(b_1 + 2k\beta\gamma a_0)}{a_1}} \int \chi(t) \mathrm{e}^{-\beta\gamma^2 k^2 t}\, \mathrm{d}t}{\alpha(a_0 + a_1 t) Q(x,t)} \end{aligned} \tag{3.15}$$

where

$$P(x,t) = I_6 \mathrm{e}^{(\frac{-a_1 z^2}{4\beta})} - \frac{\alpha I_5}{\beta} \mathrm{e}^{(\frac{-a_1 z^2}{4\beta})} \int \mathrm{e}^{(\frac{a_1 z^2}{4\beta})}\, \mathrm{d}z \tag{3.16a}$$

$$\begin{aligned} Q(x,t) = {} & \frac{\alpha I_5}{4\beta^3} \int \mathrm{e}^{(\frac{-a_1 z^2}{4\beta})} \left(\int \mathrm{e}^{(\frac{a_1 z^2}{4\beta})}\, \mathrm{d}z \right) \mathrm{d}z - \frac{I_6}{4\beta^2} \int \mathrm{e}^{(\frac{-a_1 z^2}{4\beta})}\, \mathrm{d}z + I_7 \\ & - \frac{\mathrm{e}^{\frac{-2k\gamma(b_1 + 2k\beta\gamma a_0)}{a_1}}}{2\beta} \int (I_1 + G(t))(a_0 + a_1 t)^{-1}\, \mathrm{d}t \end{aligned} \tag{3.16b}$$

and

$$z = \frac{x}{(a_0 + a_1 t)^{(1/2)}} + \frac{2(b_1 + k\beta\gamma a_0)}{a_1 (a_0 + a_1 t)^{(1/2)}} + \frac{2k\beta\gamma(a_0 + a_1 t)^{(1/2)}}{a_1}. \tag{3.16c}$$

Sub-case 2(b): $A \neq 0$. Substituting $w_3 = I_5$ along with equation (3.7*a*) in equation (3.7*b*) we get a second-order linear ODE for the variable w_2:

$$w_2'' + \frac{a_1}{2\beta} z w_2' + \frac{A a_1}{\beta} w_2 - \frac{\alpha I_5}{4\beta^3} = 0. \tag{3.17}$$

By rescaling

$$w_2 = \frac{\alpha I_5}{4\beta^2 a_1 A} + \bar{w}_2 \tag{3.18}$$

equation (3.17) can be written as

$$\bar{w}_2'' + \frac{a_1}{2\beta} z \bar{w}_2' + \frac{A a_1}{\beta} \bar{w}_2 = 0. \tag{3.19}$$

By introducing a transformation

$$\bar{w}_2(z) = y(s) \qquad s = \frac{-a_1 z^2}{4\beta} \tag{3.20}$$

equation (3.19) can be brought into the form

$$sy'' + (\tfrac{1}{2} - s)y' - Ay = 0. \tag{3.21}$$

A general solution for the equation (3.21) can be written as [20]

$$y = I_8\, {}_1F_1(A, \tfrac{1}{2}; s) + I_9 s^{(1/2)}\, {}_1F_1(A + \tfrac{1}{2}, \tfrac{3}{2}; s) \tag{3.22}$$

where I_8, I_9 are integration constants and ${}_1F_1(A, \frac{1}{2}; s)$ is a confluent hypergeometric function [21]. Using equation (3.20) we can write down the solution for $\bar{w}_2$ as

$$\bar{w}_2(z) = I_8\, {}_1F_1\left(A, \frac{1}{2}; \left(\frac{-a_1 z^2}{4\beta}\right)\right) + I_9 \left(\frac{-a_1 z^2}{4\beta}\right)^{(1/2)} {}_1F_1\left(A + \frac{1}{2}, \frac{3}{2}; \left(\frac{-a_1 z^2}{4\beta}\right)\right) \tag{3.23}$$

so that w_2 becomes

$$w_2 = \frac{\alpha I_5}{4\beta^2 a_1 A} + I_8\, {}_1F_1\left(A, \frac{1}{2}; \left(\frac{-a_1 z^2}{4\beta}\right)\right) + I_9 \left(\frac{-a_1 z^2}{4\beta}\right)^{(1/2)} {}_1F_1\left(A + \frac{1}{2}, \frac{3}{2}; \left(\frac{-a_1 z^2}{4\beta}\right)\right) \tag{3.24}$$

and w_1 takes the form

$$\begin{aligned} w_1 = -4\beta^2 \Bigg[& A I_8\, {}_1F_1\left(A + 1, \frac{3}{2}; \left(\frac{-a_1 z^2}{4\beta}\right)\right) \\ & + \frac{(2A+1)}{6} I_9 \left(\frac{-a_1 z^2}{4\beta}\right)^{(1/2)} {}_1F_1\left(A + \frac{3}{2}, \frac{5}{2}; \left(\frac{-a_1 z^2}{4\beta}\right)\right)\Bigg]. \end{aligned} \tag{3.25}$$

Using the relation (3.24) and (3.25) the solution for the equations (1.1) can be written as

$$\begin{aligned} u &= -2k\beta\gamma - \frac{2\beta R(x,t)}{(a_0 + a_1 t)^{(1/2)} S(x,t)} \\ v &= \log\left[\frac{(a_0 + a_1 t)^A \mathrm{e}^{(k^2\beta\gamma^2 t - k\gamma z (a_0 + a_1 t)^{(1/2)})}}{S(x,t)}\right]^{2\beta} \\ q &= \frac{I_5}{2\beta(a_0 + a_1 t) S(x,t)} + \frac{\int \dot{G}(t)(a_0 + a_1 t)^A\, \mathrm{d}t}{\alpha(a_0 + a_1 t)\mathrm{e}^{(\frac{2k\gamma(b_1 + 2k\beta\gamma a_0)}{a_1})} S(x,t)} \end{aligned} \tag{3.26}$$

where,

$$\begin{aligned} R(x,t) = {} & A I_8\, {}_1F_1\left(A + 1, \frac{3}{2}; \left(\frac{-a_1 z^2}{4\beta}\right)\right) \\ & + \frac{(2A+1)}{6} I_9 \left(\frac{-a_1 z^2}{4\beta}\right)^{(1/2)} {}_1F_1\left(A + \frac{3}{2}, \frac{5}{2}; \left(\frac{-a_1 z^2}{4\beta}\right)\right) \\ S(x,t) = {} & \frac{\alpha I_5}{4\beta^2 a_1 A} + I_8\, {}_1F_1\left(A, \frac{1}{2}; \left(\frac{-a_1 z^2}{4\beta}\right)\right) \\ & + I_9 \left(\frac{-a_1 z^2}{4\beta}\right)^{(1/2)} {}_1F_1\left(A + \frac{1}{2}, \frac{3}{2}; \left(\frac{-a_1 z^2}{4\beta}\right)\right) \\ & - \frac{\mathrm{e}^{(\frac{-2k\gamma(b_1 + 2k\beta\gamma a_0)}{a_1})}}{2\beta} \int (I_1 + G(t))(a_0 + a_1 t)^{A-1}\, \mathrm{d}t \end{aligned} \tag{3.27}$$

with

$$z = \frac{x}{(a_0 + a_1 t)^{(1/2)}} + \frac{2(b_1 + k\beta\gamma a_0)}{a_1(a_0 + a_1 t)^{(1/2)}} + \frac{2k\beta\gamma(a_0 + a_1 t)^{(1/2)}}{a_1}. \tag{3.28}$$

In this subcase the presence of the arbitrary function $\chi(t)$ adds more possibilities in satisfying suitable boundary conditions.

One can also carry out other subcases by assuming some of the other arbitrary constants in the infinitesimal symmetries (2.6) to be zero. However, they do not lead to any new similarity reductions.

4. Other point symmetries for the system (2.1)

In the previous two sections we have focused our attention on the determination of potential symmetries and the exploitation of new solutions for the simplified reacting mixture model. In this section, for sake of completeness, we wish to mention the existence of further point symmetries and related similarity reductions for the system (2.1) corresponding to the following forms of $f(u)$.

(1) $f(u) =$ constant $= f_0$. In this case the point symmetries turn out to be

$$\xi_1 = e_1 t + e_2 \qquad \xi_2 = a_1 \qquad \phi_1 = e_1 \qquad \phi_2 = e_1 x + e_3 \qquad \phi_3 = 0. \tag{4.1}$$

The similarity variables are

$$\begin{aligned} &z = x - \frac{e_2}{a_1}t - \frac{e_1}{2a_1}t^2 \qquad w_1 = u - \frac{e_1}{a_1}t \\ &w_2 = v - \frac{e_1^2}{6a_1^2}t^3 - \frac{e_1 e_2}{2a_1^2}t^2 - \frac{e_1}{a_1}zt - \frac{e_3}{a_1}t \qquad w_3 = q. \end{aligned} \tag{4.2}$$

The associated reduced ODE system takes the form

$$\begin{aligned} &a_1 w_2' - w_1 = 0 \\ &e_2 w_2' + \beta a_1 w_1' - \frac{a_1 w_1^2}{2} + a_1 \alpha w_3 - e_1 z - e_3 = 0 \\ &w_3' - f_0 \gamma w_3 = 0. \end{aligned} \tag{4.3}$$

(2) $f(u) =$ an arbitrary function. In this case the point symmetries turn out to be

$$\xi_1 = b_1 \qquad \xi_2 = a_1 \qquad \phi_1 = 0 \qquad \phi_2 = e_1 \qquad \phi_3 = 0. \tag{4.4}$$

The similarity variables are (with $a_1 = 1$)

$$z = x - b_1 t \qquad w_1 = u \qquad w_2 = v - e_1 t \qquad w_3 = q. \tag{4.5}$$

The associated reduced ODE system takes the form

$$\begin{aligned} &w_2' - w_1 = 0 \\ &w_2' + \frac{\beta}{b_1}w_1' - \frac{w_1^2}{2b_1} + \frac{\alpha w_3}{b_1} - \frac{e_1}{b_1} = 0 \\ &w_3' - \gamma w_3 f(w_1) = 0 \end{aligned} \tag{4.6}$$

where $f(w_1)$ is an arbitrary function of w_1.

In both cases the symmetries obtained are trivially projectable in the space $\{t, x, u, q\}$ and coincide with those of the earlier analysis made in [19]. Finally, we wish to mention that only for the case in which the system admits potential symmetry the reduced ODE system turns out to be linear (see equations (3.3) and equation (3.7)).

5. Linearization of the system (2.1)

One of the main motivations to use the potential symmetry approach for a given system is to find transformations which linearize the given nonlinear system. In this section, by employing the ideas given in [1] (theorem 6.4.1-1, p 320), we explore a transformation which linearizes the system (2.1). The aforementioned theorem gives a necessary and sufficient condition for the existence of an invertible mapping which linearizes a system admitting an infinite-parameter Lie group of transformations.

In our case, the infinitesimal operator generating the infinite parameter group of transformations is

$$\hat{X} = \left[\frac{u}{2\beta}c_1(t,x) + c_{1x}(t,x)\right]\mathrm{e}^{\frac{v}{2\beta}}\partial_u + c_1(t,x)\mathrm{e}^{\frac{v}{2\beta}}\partial_v + \left[\frac{q}{2\beta}c_1(t,x)\mathrm{e}^{\frac{v}{2\beta}} + \frac{1}{\alpha}\chi(t)\mathrm{e}^{\frac{v}{2\beta}+k\gamma x}\right]\partial_q.$$

By following the algorithm given in [1] we are able to find the following overdetermined PDE system for the unknown Φ:

$$\frac{u}{2\beta}\Phi_u + \Phi_v + \frac{q}{2\beta}\Phi_q = 0 \qquad \Phi_u = 0 \qquad \Phi_q = 0 \tag{5.1}$$

of which two independent solutions, z_1 and z_2, can be chosen as new independent variables. We choose

$$z_1 = t \qquad z_2 = x. \tag{5.2}$$

The transformation for the remaining coordinates can be obtained from the solutions of the following system:

$$\mathrm{e}^{\frac{v}{2\beta}}\left(\frac{u}{2\beta}\psi_{1u} + \psi_{1v} + \frac{q}{2\beta}\psi_{1q}\right) = 1 \qquad \psi_{1u} = 0 \qquad \psi_{1q} = 0 \tag{5.3a}$$

$$\frac{u}{2\beta}\psi_{2u} + \psi_{2v} + \frac{q}{2\beta}\psi_{2q} = 0 \qquad \mathrm{e}^{\frac{v}{2\beta}}\psi_{2u} = 1 \qquad \psi_{2q} = 0 \tag{5.3b}$$

$$\frac{u}{2\beta}\psi_{3u} + \psi_{3v} + \frac{q}{2\beta}\psi_{3q} = 0 \qquad \psi_{3u} = 0 \qquad \frac{\mathrm{e}^{k\gamma x + \frac{v}{2\beta}}}{\alpha}\psi_{3q} = 1. \tag{5.3c}$$

Particular independent solutions of equations (5.3) can be easily found to be

$$\psi_1 = -2\beta\mathrm{e}^{-\frac{v}{2\beta}} \qquad \psi_2 = u\mathrm{e}^{-\frac{v}{2\beta}} \qquad \psi_3 = \alpha q\mathrm{e}^{-\frac{v}{2\beta}+k\gamma x}. \tag{5.4}$$

As a consequence we obtain a transformation of the form

$$\begin{aligned} &z_1 = t \qquad z_2 = x \qquad \psi_1 = -2\beta\mathrm{e}^{-\frac{v}{2\beta}} \\ &\psi_2 = u\mathrm{e}^{-\frac{v}{2\beta}} \qquad \psi_3 = \alpha q\mathrm{e}^{-\frac{v}{2\beta}+k\gamma x}. \end{aligned} \tag{5.5}$$

Substituting the above transformation (5.5) into the nonlinear equation (2.1) with $f(u)$ given in equation (2.5) one gets a linear equation of the form

$$\begin{aligned} &\psi_{1z_2} - \psi_2 = 0 \\ &\psi_{1z_1} - \beta\psi_{2z_2} - \mathrm{e}^{k\gamma z_2}\psi_3 = 0 \\ &\psi_{3z_2} = 0. \end{aligned} \tag{5.6}$$

It is a simple matter to verify that, as a consequence, the system (1.1) is linearized.

It is worthwhile noticing that it is possible to reduce the system (5.6) to the following linear second-order PDE:

$$\psi_{1t}(t,x) - \beta\psi_{1xx}(t,x) = \mu(t)\mathrm{e}^{k\gamma x} \tag{5.7}$$

where we have taken (5.2) into acount and $\mu(t)$ is an arbitrary function of t.

6. Conclusions

In this paper we have carried out a detailed group theoretical analysis of a simplified nonlinear model for a binary reacting unimolecular mixture using the potential symmetries approach. A detailed investigations of the non-potential symmetries has been carried out in [19]. The potential symmetries which we have found here allow us to determine new wide classes of exact solutions. They may serve, as usual, for a benchmark test of a large numerical scheme devised to solve the system in a realistic case. However, for the classes of solutions found in the previous sections, the presence of arbitrary functions enlarge to the large extent the possibility to find solutions satisfying realistic initial or boundary conditions. Finally, taking into account that the auxiliary system (2.1) admits an infinite parameter Lie group of transformations we linearize the system following the algorithm suggested by Bluman and Kumei. This result is very useful for a qualitative analysis of system (1.1).

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